Special Functions Supported by EasyCalc

Rafael R. Sevilla

February 24, 2001

EasyCalc now supports a large number of "special functions," which should be especially useful for those involved in mathematical physics, boundary value problems, and statistics. Most of these functions will only be available if you apply the flag `--enable-specfun` to the `configure` script for EasyCalc. These functions are for the most part based on approximations given in Chapter 6 of *Numerical Recipes* [3], and the public domain Cephes code available from Netlib. More information on all of these special functions is available from Abramowitz and Stegun [1]. Discussions of the theory, special properties, and applications of these functions can be found in [2] and in [4].

Summary of Special Functions

Function Name	Name in EasyCalc
Euler Gamma	`gamma(z)`
Beta	`beta(z:w)`
Incomplete Gamma	`igamma(a:x)`
Error	`erf(x)`
Complementary Error	`erfc(x)`
Incomplete Beta	`ibeta(a:b:x)`
Bessel 1st Kind	`besselj(n:x)`
Bessel 2nd Kind	`bessely(n:x)`
Mod. Bessel 1st Kind	`besseli(n:x)`
Mod. Bessel 2nd Kind	`besselk(n:x)`
Inc. Elliptic Integral 1st kind	`elli1(m:phi)`
Inc. Elliptic Integral 2nd kind	`elli2(m:phi)`
Comp. Elliptic Integral 1st kind	`ellc1(m)`
Comp. Elliptic Integral 2nd kind	`ellc2(m)`
Jacobi sn	`sn(m:u)`
Jacobi cn	`cn(m:u)`
Jacobi dn	`dn(m:u)`

The Euler Gamma Function

The Euler gamma function is the only function here described that is available whether or not you set `--enable-specfun`, but if that flag is not set then you will be able to use the gamma function only for real arguments; to be able to use it for complex arguments you will need to set that flag. It is used by the factorial function to compute non-integral values for the factorial as well. The Euler gamma function is defined by the integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{x-1} dt \tag{1}$$

You can call the gamma function as `gamma(`*`arg`*`)`. Note that this function is automatically called implicitly whenever you do `fact(`*`arg`*`)` where *`arg`* is not an integer. The Euler gamma function should not be evaluated for negative integer arguments.

The Beta Function

The Beta function is defined by

$$B(z,w) = B(w,z) = \int_0^1 t^{z-1}(1-t)^{w-1} dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{2}$$

Call it as `beta(`*`z`*`:`*`w`*`)`. It just uses the gamma function to calculate it directly. Do not call it with negative values for z and/or w.

The Incomplete Gamma Function

The incomplete gamma function is defined by

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \tag{3}$$

You can access it with `igamma(a:x)`. Values of $a \leq 0$ or for $x < 0$ are invalid and produce an error.

The Error Function and the Complementary Error Function

These functions are defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{4}$$

and

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{5}$$

Call them as `erf(x)` and `erfc(x)`.

The Incomplete Beta Function

This function is defined by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt \tag{6}$$

where $B(a,b)$ is the Beta Function (Eq. 2) and can be called as `ibeta(a:b:x)`. Both a and b must be greater than 0, and $0 \le x \le 1$. Do not use it for values other than these allowed.

Bessel Functions of the First and Second Kinds

The Bessel functions of the first kind, $J_\nu(x)$ arise as solutions to the differential equation:

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \tag{7}$$

and is defined by the series representation

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \tag{8}$$

You can evaluate it with EasyCalc by doing `besselj(nu:x)`. The order ν of the Bessel function is restricted to nonnegative integers in this version.

The Bessel functions of the second kind $Y_\nu(x)$ are the second linearly independent solutions to Eq. 7. For ν *not* an integer, $Y_\nu(x)$ can be expressed in terms of the Bessel functions of the first kind as:

$$Y_\nu(x) = \frac{J_\nu(x)cos(\nu\pi) - J_{-\nu}(x)}{sin(\nu\pi)} \tag{9}$$

but it produces correct results in the limit as ν approaches an integer. You can evaluate this by doing `bessely(nu:x)`. Again, the order ν is restricted to integers, and note that all the Bessel functions of the second kind possess singularities at zero, so don't try to evaluate it there.

Modified Bessel functions of the First and Second Kinds

These functions, $I_\nu(x)$ and $K_\nu(x)$, arise as the linearly independent solutions to the differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0 \tag{10}$$

and are the regular Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ evaluated for purely imaginary arguments:

$$I_\nu(x) = (-i)^\nu J_\nu(ix) \tag{11}$$

and

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iY_\nu(ix)] \tag{12}$$

They are evaluated by `besseli(nu:x)` and `besselk(nu:x)`. Again the order ν is restricted to integers, and the K_ν functions have a singularity at zero.

Elliptic Integrals

The incomplete elliptic integral of the first kind is defined as follows:

$$F(\phi|m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}} \tag{13}$$

with eccentricity/modulus m and amplitude ϕ. You can compute this function by entering `elli1(m:phi)`. The complete elliptic integral of the first kind:

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}} \tag{14}$$

and may be computed as `ellc1(m)`. The incomplete elliptic integral of the second kind:

$$E(\phi|m) = \int_0^\phi \sqrt{1 - m^2 \sin^2 \theta} d\theta \tag{15}$$

may be calculated with `elli2(m:phi)`, while the complete elliptic integral of the second kind

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m^2 \sin^2 \theta} d\theta \tag{16}$$

may be computed as `ellc2(m)`. All of these elliptic functions can be computed only with an eccentricity m between 0 and 1. Values outside this are considered out of range.

Jacobian Elliptic Functions

These functions are inverses of the elliptic integral of the first kind $F(\phi|m)$. The Jacobian elliptic function $\mathrm{sn}(u|m)$ is defined as

$$\mathrm{sn}(F(\phi|m)|m) = \sin \phi \tag{17}$$

The other two functions, sn and dn, can be defined by the relations

$$\mathrm{cn}(F(\phi|m)|m) = \cos \phi \tag{18}$$

and

$$\mathrm{dn}(F(\phi|m)|m) = \sqrt{1 - m^2 \sin^2 \phi} \tag{19}$$

or equivalently,

$$\mathrm{sn}^2(u|m) + \mathrm{cn}^2(u|m) = 1 \tag{20}$$

and

$$m^2\mathrm{sn}^2(u|m) + \mathrm{dn}^2(u|m) = 1 \tag{21}$$

These three functions may be calculated as `sn(`*m*`:`*u*`)`, `cn(`*m*`:`*u*`)` and `dn(`*m*`:`*u*`)`. As with the elliptic integrals, the eccentricity m may only be between 0 and 1.

References

[1] Abramowitz, Milton, and Irene A. Stegun. *Handbook of Mathematical Functions*, Applied Mathematics Series, vol. 55. Washington: National Institute of Standards and Technology, 1968.

[2] Andrews, Larry C. *Special Functions of Mathematics for Engineers.* New York: McGraw-Hill, Inc, 1992.

[3] Press, William H., Brian P. Flannery, Saul A. Teukolsky, and William K. Vetterling. *Numerical Recipes in FORTRAN.* Cambridge: Cambridge University Press, 1986.

[4] Whittaker, E. T., and G. N. Watson. *A Course of Modern Analysis.* Cambridge: Cambridge University Press, 1927.